Harmonizing Classical and Quantum Mechanics: A semi-classical approach to quantum systems

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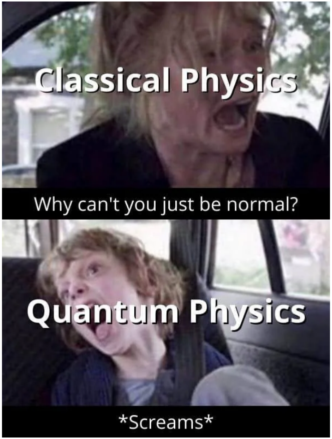
Motivation


Classical Physics
Why can't you just be normal?
Quantum Physics
Screams

1. Introduction
   - Classical Mechanics: An overview
   - Quantum Mechanics: An overview
   - Classical vs Quantum
2. Density Operators
3. Weyl Transform
4. Wigner Functions
5. Classical limit
6. Summary

Hamiltonian Dynamics

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For conservative systems,

$$H(\mathrm{q}, \mathrm{p}, t) = T(\mathrm{q}, \dot{\mathrm{q}}) + V(\mathrm{q}, t)$$

Hamiltonian Dynamics (continued)

- Time Evolution :

$$\dot{q}_k = \frac{\partial H}{\partial p_k} = \{q_k, H\}$$
$$\dot{p}_k = -\frac{\partial H}{\partial q_k} = \{p_k, H\}$$

where, $\{F, G\} = \sum_i \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$ is called the Poisson bracket.

Phase space formulation

- Phase space coordinates : (q, p)
- Time evolution $\implies$ Trajectories in phase space
- **An Example :** 1-D Simple Harmonic Oscillator

$$H = \frac{1}{2m}p^2 + \frac{1}{2}kq^2$$

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- **Inner product :**

$$\langle\phi|\psi\rangle = \begin{pmatrix} \phi_1^* & \phi_2^* & \cdot\cdot & \phi_n^* \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \psi_n \end{pmatrix} = \sum_{k=1}^{n} \phi_k^* \psi_k$$

Linear Operators

- $\hat{A}\left|\psi\right\rangle = \left|\phi\right\rangle$
- Linearity :
  - $\hat{A}(\left|\psi_1\right\rangle + \left|\psi_2\right\rangle) = \hat{A}\left|\psi_1\right\rangle + \hat{A}\left|\psi_2\right\rangle$
  - $\hat{A}(c\left|\psi\right\rangle) = c\hat{A}\left|\psi\right\rangle$
- Linear operator $\longleftrightarrow$ Matrix
  In the basis $\{\left|\phi_1\right\rangle, \left|\phi_2\right\rangle, \cdots, \left|\phi_n\right\rangle\}$, the matrix elements of $\hat{A}$ are -

$$A_{ij} = \left\langle\phi_i\right|\hat{A}\left|\phi_j\right\rangle$$

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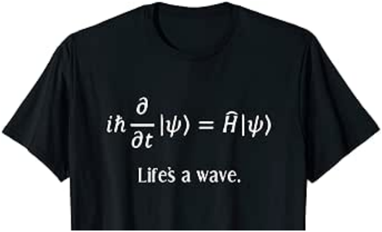

***Commonly found on t-shirts.*

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Classical vs Quantum : Summary

	Classical Mechanics	Quantum Mechanics
State of the system	Phase space variables	Ket states
Observables	Functions of phase space variables	Self-adjoint operators
Time Evolution	Trajectories in phase space (Hamilton's equations)	Schrödinger equation

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Time evolution of density matrices

Liouville-von Neumann equation :

$$i\hbar\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = [\hat{H}, \hat{\rho}]$$

Proof :

$$\text{Schrödinger equation : } i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

$$i\hbar\frac{d}{dt}\rho(t) = \left(i\hbar\frac{d}{dt}|\psi(t)\rangle\right)\langle\psi(t)| + |\psi(t)\rangle\left(i\hbar\frac{d}{dt}\langle\psi(t)|\right)$$

$$i\hbar\frac{d}{dt}\rho(t) = H\rho(t) - \rho(t)H = [H, \rho(t)]$$

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- Weyl transform of an operator $\hat{A}$ is defined as -

$$\tilde{A}(x,p) = \int dy\ exp\left(-\frac{ipy}{\hbar}\right)\langle x+\frac{y}{2} \mid \hat{A} \mid x-\frac{y}{2}\rangle$$
$$= \int du\ exp\left(\frac{ixu}{\hbar}\right)\langle p+\frac{u}{2} \mid \hat{A} \mid p-\frac{u}{2}\rangle$$

Examples and properties

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- $$Tr[\hat{A}\hat{B}] = \frac{1}{h}\int\int \tilde{A}(x,p)\tilde{B}(x,p)\ dx\ dp$$

Weyl-Groenewold-Moyal star product

- Let us denote the weyl transform of $\hat{A}$ as $\mathcal{W}(\hat{A})$. Then,

$$\mathcal{W}(\hat{A}\hat{B}) = \mathcal{W}(\hat{A}) \star \mathcal{W}(\hat{B})$$

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- It is non-commutative and associative.
- Moyal bracket OR $\star$-commutator : $[f,g]_\star = f \star g - g \star f$

Describing state of the system

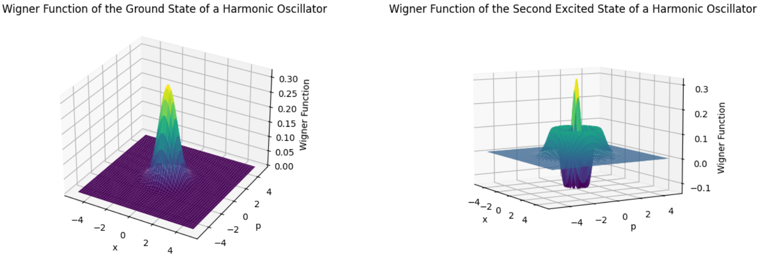
- Wigner function : Analogue of wave function
- Weyl transform of density operator

$$W(x,p) = \frac{1}{h}\int e^{-ipy/\hbar}\,\langle x+\frac{y}{2}|\hat{\rho}|x-\frac{y}{2}\rangle\, dy,$$

$$\implies W(x,p) = \frac{1}{h}\int e^{-ipy/\hbar}\,\langle x+\frac{y}{2}\,|\psi\rangle\,\langle\psi|\,x-\frac{y}{2}\rangle\, dy,$$

$$\implies W(x,p) = \frac{1}{h}\int e^{-ipy/\hbar}\,\psi\left(x+\frac{y}{2}\right)\,\psi^*\left(x-\frac{y}{2}\right)\, dy.$$

Examples



(a) Ground state (b) Second excited state

Figure 1: Wigner functions for different states of the quantum harmonic oscillator

Properties

- **Real valued function**
- **Quasi-probability distribution**
- **Normalization** :

$$\int \int W(x, p)\, dx\, dp = 1$$

- **Marginal distributions** :

$$\int W(x, p)\, dp = \psi^*(x)\psi(x) = |\psi(x)|^2$$

$$\int W(x, p)\, dx = \psi^*(p)\psi(p) = |\psi(p)|^2$$

Properties (continued)

- **Expectation values** :

$$\langle \hat{A} \rangle = \int \int W(x,p)\, \tilde{A}(x,p)\, dx\, dp$$

- **Orthogonality** :

$$\int \int W_a(x,p)\, W_b(x,p)\, dx\, dp = \frac{1}{h} |\langle \psi_a | \psi_b \rangle|^2$$

- **Bound on the distribution** :

$$|W(x,p)| \leq \frac{2}{h}$$

Time evolution of wigner function

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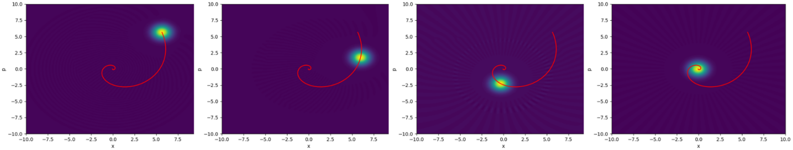

Figure 2: Time evolution of Wigner function for a dissipative quantum system

Classical limit

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  - Wigner function being negative in regions of phase space.
  - Dependence of spread of distribution on $\hbar$.
- $$f \star g = fg + \frac{i\hbar}{2}\{f, g\}$$
  $$-\frac{\hbar^2}{8}\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 g}{\partial p^2} + \frac{\partial^2 f}{\partial p^2}\frac{\partial^2 g}{\partial x^2} - 2\frac{\partial^2 f}{\partial x \partial p}\frac{\partial^2 g}{\partial x \partial p}\right) + \mathcal{O}(\hbar^3)$$

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- It naturally incorporates mixed states in it.
- *"It is the way in which physical results are extracted, using not only the Wigner function, but also the Weyl transform of the desired operator and the star product algebra, which accounts for the quantum behaviour."*[1]
- *"In some other part of the universe, wigner-weyl approach might be the method of doing quantum mechanics discovered first."*[2]

[1] W. B. Case, “Wigner functions and weyl transforms for pedestrians,” *American Journal of Physics*, vol. 76, no. 10, pp. 937–946, 2008.

[2] J. Hancock, M. Walton, and B. Wynder, “Quantum mechanics another way,” *European journal of physics*, vol. 25, no. 4, p. 525, 2004.

[3] H. Goldstein, *Classical mechanics*. Pearson Education India, 2011.

[4] R. Shankar, *Principles of quantum mechanics*. Springer Science & Business Media, 2012.

Thanks For Your Attention!!!

Any questions?